Purely Affine Formulation of F(R) Gravity

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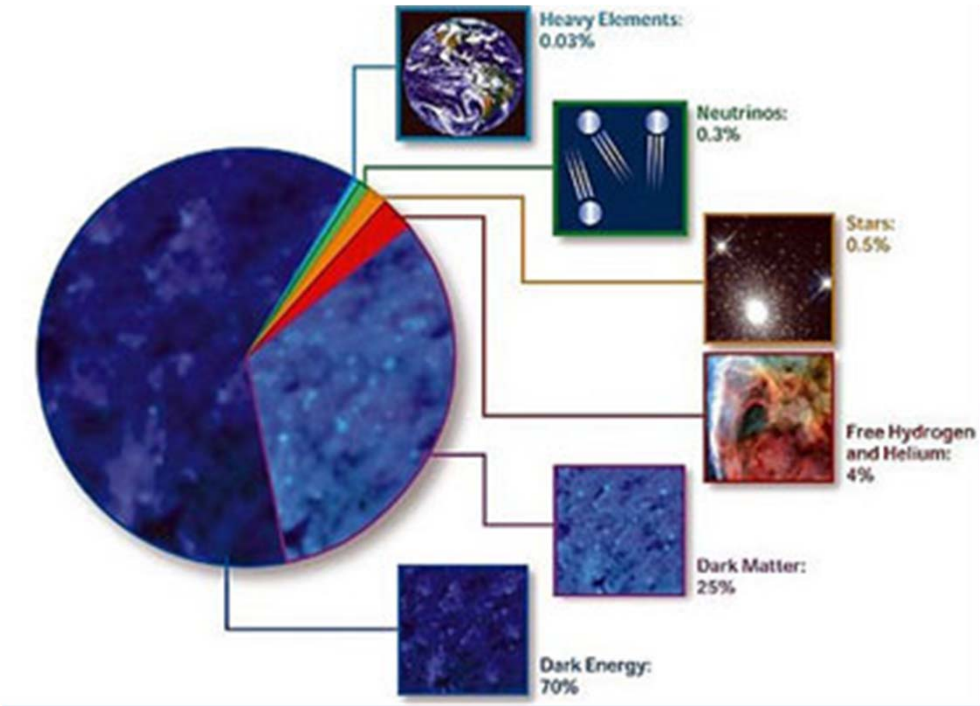
CURRENT COSMIC ACCELERATION



We are living in an accelerating universe

$L=F(R)=R-R_0^2/R$

D. N. Vollick, Phys. Rev. D **68**, 063510 (2003)
S. M. Carroll et al., Phys. Rev. D **70**, 043528 (2004)

EXPLANATION

Cosmological constant

ΛCDM model

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} + \Lambda g_{\mu\nu}$$

Agrees with observations

$$\Lambda \approx 10^{-52} m^{-2}$$

Other possibilities include:

- *Dark energy*
- *Alternative gravity theories*

(such as **F(R) gravity** that also can explain inflation)

3 FORMULATIONS OF GR

PURELY AFFINE (Einstein–Eddington)

- Torsionless connection Γ is a dynamical variable
- Lagrangian density L depends on the symmetric part of the Ricci tensor R(Γ)
- The metric tensor g = the derivative of L with respect to R

METRIC–AFFINE (Einstein–Palatini)

- Both g and torsionless Γ are independent variables
- L is proportional to gR(Γ)

PURELY METRIC (Einstein–Hilbert)

- g is a dynamical variable
- Γ is the Christoffel connection of g
- L is proportional to gR(g)

This equivalence works for Lagrangians that depend also on antisymmetrized Ricci, segmental curvature, and torsion tensor.

All 3 formulations of gravitation are dynamically equivalent

M. Ferraris and J. Kijowski, Gen. Relativ. Gravit. **14**, 165 (1982)

PURELY AFFINE GRAVITY

Metric density $\mathrm{g}^{\mu\nu} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial P_{\mu\nu}}$ $P_{\mu\nu} = R_{(\mu\nu)}$

Metric tensor $g^{\mu\nu} \equiv \frac{\mathrm{g}^{\mu\nu}}{\sqrt{-\det \mathrm{g}^{\rho\sigma}}}$ $\det(\mathrm{g}^{\mu\nu}) < 0$

$\rightarrow$ algebraic relation R(g)

Define

$$\mathrm{h}^{\mu\nu} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial Q_{\mu\nu}} \qquad Q_{\mu\nu} = R^{\rho}{}_{\rho\mu\nu} = \Gamma^{\,\rho}_{\rho\nu,\mu} - \Gamma^{\,\rho}_{\rho\mu,\nu}$$

segmental curvature

$$\Pi^{\mu}{}_{\rho}{}^{\nu} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial \Gamma^{\,\rho}_{\mu\nu}}$$

hypermomentum

Field equations from purely affine variational principle

(variation of action with respect to $\Gamma^{\,\rho}_{\mu\,\nu}$)

$\rightarrow$ differential relation Γ(g) $\rightarrow$ differential equation for g

LAGRANGE FIELD EQUATIONS

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^*\Gamma^{\,\mu}_{\sigma\rho}\mathrm{g}^{\sigma\nu} + {}^*\Gamma^{\,\nu}_{\rho\,\sigma}\mathrm{g}^{\mu\sigma} - {}^*\Gamma^{\,\sigma}_{\sigma\,\rho}\mathrm{g}^{\mu\nu}$$

$$= \Pi^{\mu}{}_{\rho}{}^{\nu} - \frac{1}{3}\Pi^{\mu}{}_{\sigma}{}^{\sigma}\delta^{\nu}_{\rho} + 2\mathrm{h}^{\nu\sigma}{}_{,\sigma}\delta^{\mu}_{\rho} - \frac{2}{3}\mathrm{h}^{\mu\sigma}{}_{,\sigma}\delta^{\nu}_{\rho}$$

yield

$${}^*\Gamma^{\,\rho}_{\mu\nu} = \Gamma^{\,\rho}_{\mu\nu} + \tfrac{2}{3}\delta^{\rho}_{\mu}S_{\nu}$$

$$\mathrm{h}^{\sigma\nu}{}_{,\sigma} = \mathrm{j}^{\nu} \equiv \frac{1}{8}\Pi^{\sigma}{}_{\sigma}{}^{\nu}$$ ← looks like 2nd Maxwell equation

conserved current

If L is independent of Q then FE constrain how L depends on Γ, unless:
- L does not depend on Γ
- L depends also on other parts of curvature, torsion or nonmetricity
- Torsion tensor is constrained by a projectively non-invariant condition, e.g., its trace vanishes

NJP, Ann. Fond. Louis de Broglie **32**, 335 (2007)

ANALOGY WITH CLASSICAL MECHANICS

Generalized coordinate	q^i	$\Gamma_{\mu\ \nu}^{\ \rho}$
Generalized velocity	$\dot{q}^i$	$P_{\mu\nu}$
Lagrangian	$\mathfrak{L}(\Gamma_{\mu\nu}^{\ \rho}, P_{\mu\nu})$	$L(q^i, \dot{q}^i)$
Canonical momentum	$p^i = \dfrac{\partial L}{\partial \dot{q}^i}$	$\mathrm{g}^{\mu\nu} \equiv -2\kappa \dfrac{\partial \mathfrak{L}}{\partial P_{\mu\nu}}$
Generalized force	$f^i = \dfrac{\partial L}{\partial q^i}$	$\Pi^{\mu\ \nu}_{\ \rho} \equiv -2\kappa \dfrac{\partial \mathfrak{L}}{\partial \Gamma_{\mu\nu}^{\ \rho}}$

(What in GR corresponds to Poisson brackets and canonical transformations?)

J. Kijowski, Gen. Relativ. Gravit. **9**, 857 (1978)

EQUIVALENCE OF AFFINE AND METRIC GRAVITY

Define Hamiltonian via **Legendre** transformation with respect to P (F&K)

$$\mathfrak{H}(\Gamma^{\rho}_{\mu\nu}, \mathrm{g}^{\mu\nu}) = \mathfrak{L} - \frac{\partial \mathfrak{L}}{\partial P_{\mu\nu}} P_{\mu\nu} = \mathfrak{L} + \boxed{\frac{1}{2\kappa} \mathrm{g}^{\mu\nu} P_{\mu\nu}} \qquad H = \boxed{p_i \dot{q}_i} - L$$

1st Hamilton equation
→ Einstein equations

$$P_{\mu\nu} = 2\kappa \frac{\partial \mathfrak{H}}{\partial \mathrm{g}^{\mu\nu}} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$\mathfrak{H}$ corresponds to metric-affine Lagrangian for matter $\mathcal{L}$

2nd Hamilton equation → FE

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}$$

Metric-affine Lagrangian for the gravitational field is a **Legendre** term **pv**
→ **automatically linear in R**

Metric–affine → purely metric: Legendre transformation with respect to Γ

F(R) LAGRANGIAN

Reverse **Legendre** transformation from H to L (with respect to g)

GR: $\mathfrak{L} = \mathfrak{H} - \frac{\partial \mathfrak{H}}{\partial \mathrm{g}^{\mu\nu}} \mathrm{g}^{\mu\nu} = \mathfrak{H} - \frac{1}{2\kappa} \mathrm{g}^{\mu\nu} P_{\mu\nu}$

F(R) metric-affine Lagrangian $\mathcal{L}_g = -\frac{1}{2\kappa}\sqrt{-g}F(R)$

Einstein equations $F' P_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(F - RF') = 2\kappa \frac{\partial \mathfrak{H}}{\partial \mathrm{g}^{\mu\nu}}$

reverse Legendre transformation →

$$\delta \mathfrak{L} = -\frac{1}{2\kappa} \Pi^{\mu}{}_{\rho}{}^{\nu} \delta \Gamma^{\rho}_{\mu\nu} - \frac{1}{2\kappa}(F' - RF'')\mathrm{g}^{\mu\nu} \delta P_{\mu\nu}$$
$$- \frac{1}{2\kappa}\left(\frac{1}{2}(F - RF')\mathrm{g}^{\mu\nu} + RF'' P^{\mu\nu}\sqrt{-g}\right)\delta g_{\mu\nu}$$

L is purely affine if $\frac{1}{2}(F - RF')g^{\mu\nu} + RF'' P^{\mu\nu} = 0 \ \rightarrow \ F(R) = \alpha R$

Einstein-Hilbert

EINSTEIN FRAME

Metric-affine Lagrangian $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{MA} = \mathfrak{H} - \frac{1}{2\kappa}\sqrt{-g}F(R)$

Is dynamically equivalent to scalar-tensor Lagrangian

$$\mathcal{L}_{ST} = \mathfrak{H}(\Gamma^{\,\rho}_{\mu\nu}, g_{\mu\nu}) - \frac{1}{2\kappa}\sqrt{-g}(F(\phi) + F^{\cdot}(\phi)(R - \phi))$$

Define **conformally** transformed metric tensor $\tilde{g}_{\mu\nu} = F^{\cdot}(\phi)g_{\mu\nu}$
(transform from Jordan to Einstein frame)

Then $\mathcal{L}_{ST} = \tilde{\mathfrak{H}}(\Gamma^{\,\rho}_{\mu\nu}, \tilde{g}_{\mu\nu}, \phi) - \frac{1}{2\kappa}P_{\mu\nu}\tilde{g}^{\mu\nu}$ $\tilde{\mathfrak{H}} = \mathfrak{H} - \frac{1}{2\kappa}(F^{\cdot})^{-2}\sqrt{-\tilde{g}}(F - \phi F^{\cdot})$

GR

Einstein equations $P_{\mu\nu} = 2\kappa\frac{\partial\tilde{\mathfrak{H}}}{\partial\tilde{\mathrm{g}}^{\mu\nu}}$

reverse
Legendre
transformation
$\rightarrow$

$$\tilde{\mathfrak{L}} = \tilde{\mathfrak{H}} - \frac{\partial\tilde{\mathfrak{H}}}{\partial\tilde{\mathrm{g}}^{\mu\nu}}\tilde{\mathrm{g}}^{\mu\nu} = \tilde{\mathfrak{H}} - \frac{1}{2\kappa}\tilde{\mathrm{g}}^{\mu\nu}P_{\mu\nu}$$

$$d\tilde{\mathfrak{L}} = \frac{\partial\tilde{\mathfrak{H}}}{\partial\Gamma^{\,\rho}_{\mu\nu}}d\Gamma^{\,\rho}_{\mu\nu} + \frac{\partial\tilde{\mathfrak{H}}}{\partial\phi}d\phi - \frac{1}{2\kappa}\tilde{\mathrm{g}}^{\mu\nu}\,dP_{\mu\nu} \rightarrow \tilde{\mathfrak{L}} = \tilde{\mathfrak{L}}(\Gamma^{\,\rho}_{\mu\nu}, P_{\mu\nu}, \phi)$$ **L purely affine**

SUMMARY

• One cannot construct a dynamically equivalent, purely affine Lagrangian from a metric-affine or purely metric, nonlinear F(R) Lagrangian. F(R) has no purely affine formulation unless F(R)=R (GR).

• This equivalence is restored if one uses the Einstein-frame metric tensor as the generalized momentum in the Legendre transformation from the metric-affine (Hamiltonian) to the purely affine (Lagrangian) dynamics.

• The peculiar behavior of GR, among relativistic theories of gravity, with respect to the equivalence between the purely affine, metric-affine and purely metric pictures may indicate that F(R) gravity is just a different description of GR with a scalar field, where the scalar degree of freedom artificially enters the curvature part of the Lagrangian.

NJP, Int. J. Mod. Phys. A **23**, 1891 (2008)