Homothetic curvature and geometrization of electromagnetism

Nikodem J. Popławski

Department of Physics, Indiana University, Bloomington, IN

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ELECTROMAGNETIC FIELD IN AFFINE CONNECTION

• Enough degrees of freedom in connection to describe gravitational and electromagnetic fields (classically).

• Affine formulation allows for elegant unification of both fields.
- Purely affine: M. Ferraris, J. Kijowski, GRG (1982)
- Metric-affine: V. N. Ponomarev, Y. N. Obukhov, GRG (1982)

Gravitational field – symmetric part of the Ricci tensor of the connection

$$P_{\mu\nu} = R_{(\mu\nu)}$$

EM field – the tensor of **homothetic curvature Q** (which is a curl)

$$Q_{\mu\nu} \doteq R^{\rho}{}_{\rho\mu\nu} = \Gamma^{\ \rho}_{\rho\,\nu,\mu} - \Gamma^{\ \rho}_{\rho\,\mu,\nu} = -\frac{1}{2}(N^{\rho}{}_{\rho\nu,\mu} - N^{\rho}{}_{\rho\mu,\nu})$$

$N_{\mu\nu\rho} = g_{\mu\nu;\rho}$ – nonmetricity tensor

More natural than relating EM potential A to metric (Weyl, Kaluza, Einstein-Straus-Schrödinger), because both A and Γ have the same purpose: to correct ordinary derivative into covariant derivative.

LAGRANGIAN AND FIELD EQUATIONS FROM δg

Lagrangian: $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_Q + \mathcal{L}_m$

Gravitational part: $\mathcal{L}_g = -\frac{1}{2\kappa} R_{\mu\nu} \mathrm{g}^{\mu\nu}$

Homothetic part: $\mathcal{L}_Q = \frac{\alpha^2}{4} \sqrt{-g} Q_{\mu\nu} Q^{\mu\nu}$ $\quad \mathrm{h}^{\mu\nu} = -2\kappa \frac{\partial \mathfrak{L}}{\partial Q_{\mu\nu}}$

Matter part: energy-momentum $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}$

hypermomentum (Hehl) $\Pi^{\mu}{}_{\rho}{}^{\nu} = -2\kappa \frac{\delta \mathcal{L}_m}{\delta \Gamma^{\rho}_{\mu\nu}}$

Metric-affine Einstein equations:

$$R_{(\mu\nu)} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} - \kappa\alpha^2 \left(\frac{1}{4} Q_{\rho\sigma} Q^{\rho\sigma} g_{\mu\nu} - Q_{\mu\rho} Q_{\nu}{}^{\rho} \right)$$

FIELD EQUATIONS FROM δΓ

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^*\Gamma^{\ \mu}_{\sigma\ \rho}\mathrm{g}^{\sigma\nu} + {}^*\Gamma^{\ \nu}_{\rho\ \sigma}\mathrm{g}^{\mu\sigma} - {}^*\Gamma^{\ \sigma}_{\sigma\ \rho}\mathrm{g}^{\mu\nu}$$
$$= \Pi^{\mu\ \nu}_{\ \rho} - \frac{1}{3}\Pi^{\mu\ \sigma}_{\ \sigma}\delta^{\nu}_{\rho} + 2\mathrm{h}^{\nu\sigma}{}_{,\sigma}\delta^{\mu}_{\rho} - \frac{2}{3}\mathrm{h}^{\mu\sigma}{}_{,\sigma}\delta^{\nu}_{\rho}$$

Metric Einstein eqs.

↑ + M-A Einstein

gives

Γ(g,Π) → differential R(g,Π)

$$\mathrm{h}^{\nu\sigma}{}_{,\sigma} = \mathrm{j}^{\nu} \equiv \frac{1}{8}\Pi^{\sigma\ \nu}_{\ \sigma}$$

← looks like 2nd pair of **Maxwell** eqs.

$$\mathrm{h}^{\mu\nu} = -\kappa\alpha^2\sqrt{-g}Q^{\mu\nu}$$

$${}^*\Gamma^{\ \rho}_{\mu\ \nu} = \Gamma^{\ \rho}_{\mu\ \nu} + \tfrac{2}{3}\delta^{\rho}_{\mu}S_{\nu}$$

- The current density is conserved even if $\mathfrak{L}$ depends on $R_{[\mu\nu]}$
- If $\mathfrak{L}$ is independent of Q then field equations constrain how $\mathfrak{L}$ depends on Γ – unphysical – EM represented by Q **replaces** this constraint by Maxwell equations.

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SOLUTION OF METRIC-AFFINE FIELD EQUATIONS

$$^{*}\Gamma_{\mu\,\nu}^{\,\rho} = \left\{{}^{\,\rho}_{\mu\,\nu}\right\}_g + V^{\rho}{}_{\mu\nu}$$

ArXiv: 0705.0351 [gr-qc]

$$Q_{\mu\nu} = -\frac{8}{3}(S_{\nu,\mu} - S_{\mu,\nu}) + V^{\rho}{}_{\rho\nu,\mu} - V^{\rho}{}_{\rho\mu,\nu}$$

$$R_{\mu\nu} = R^{(g)}_{\mu\nu} - \tfrac{4}{3}S_{[\nu:\mu]} + 2V^{\rho}{}_{\mu[\nu:\rho]} + V^{\sigma}{}_{\mu\nu}V^{\rho}{}_{\sigma\rho} - V^{\sigma}{}_{\mu\rho}V^{\rho}{}_{\sigma\nu}$$

V linear in Π

$$V^{\rho}{}_{(\mu\nu)} = \frac{1}{2}(\Delta^{\rho}{}_{\nu}{}^{\sigma}\mathrm{g}_{\mu\sigma} + \Delta^{\rho}{}_{\mu}{}^{\sigma}\mathrm{g}_{\nu\sigma} - \Delta^{\alpha}{}_{\gamma}{}^{\beta}\mathrm{g}_{\mu\alpha}\mathrm{g}_{\nu\beta}\mathrm{g}^{\rho\gamma})$$

$$V^{\rho}{}_{[\mu\nu]} = \frac{1}{2}(\Omega_{\nu}{}^{\rho\sigma}\mathrm{g}_{\mu\sigma} - \Omega_{\mu}{}^{\rho\sigma}\mathrm{g}_{\nu\sigma} - \Omega_{\gamma}{}^{\alpha\beta}\mathrm{g}_{\mu\alpha}\mathrm{g}_{\nu\beta}\mathrm{g}^{\rho\gamma})$$

$$\Sigma^{\mu}{}_{\rho}{}^{\nu} = \Pi^{(\mu}{}_{\rho}{}^{\nu)} - \tfrac{1}{3}\delta^{(\mu}_{\rho}\Pi^{\nu)}{}_{\sigma}{}^{\sigma} - \tfrac{1}{6}\Pi^{\sigma}{}_{\sigma}{}^{(\mu}\delta^{\nu)}_{\rho}$$

$$\Delta^{\mu}{}_{\rho}{}^{\nu} = \Sigma^{\mu}{}_{\rho}{}^{\nu} - \tfrac{1}{2}\Sigma^{\alpha}{}_{\rho}{}^{\beta}g_{\alpha\beta}g^{\mu\nu}$$

$$\Omega_{\rho}{}^{\mu\nu} = \Pi^{[\mu}{}_{\rho}{}^{\nu]} - \tfrac{1}{3}\Pi^{[\sigma}{}_{\sigma}{}^{\nu]}\delta^{\mu}_{\rho} + \tfrac{1}{3}\Pi^{[\sigma}{}_{\sigma}{}^{\mu]}\delta^{\nu}_{\rho}$$

If no sources (Π=0)
– nonmetricity trace
is ~ to torsion vector

MATTER LAGRANGIAN

Matter – four-velocity field U enters Lagrangian in constraint parametrizing length of world line (Taub):

$$\mathcal{L}_u = -\sqrt{-g}\Lambda(u^\mu u_\mu - 1)$$

U is not variational variable; action is varied with respect to g, Γ and Λ.

U couples to trace of nonmetricity:

ArXiv: 0802.4453 [gr-qc]

$$\mathcal{L}_N = \sqrt{-g}kN^\rho{}_{\rho\mu}u^\mu \to \Pi^\mu{}_\rho{}^\nu = 4\kappa\sqrt{-g}k\delta^\mu_\rho u^\nu \to V^\rho{}_{\mu\nu} = 0\,!$$

Connection = Christoffel + torsion vector (~ trace of nonmetricity). Same as without sources.

Metric Einstein equations:

$$\kappa^{-1}G_{\mu\nu} = -\alpha^2\left(\frac{1}{4}Q_{\rho\sigma}Q^{\rho\sigma}g_{\mu\nu} - Q_{\mu\rho}Q_\nu{}^\rho\right) + 2\Lambda u_\mu u_\nu + kN^\rho{}_{\rho(\mu}u_{\nu)} - \frac{k}{2}g_{\mu\nu}N^\sigma{}_{\sigma\rho}u^\rho$$

Lagrange multiplier Λ ~ energy density

$$\Lambda = \frac{\rho_m}{2}$$

pressure = 0

GEOMETRIZATION OF ELECTROMAGNETISM

$$Q_{\mu\nu} = \frac{i}{\alpha} F_{\mu\nu} \;\rightarrow\; N^{\rho}{}_{\rho\mu} = -\frac{2i}{\alpha} A_{\mu}$$

Trace of nonmetricity geometrically represents EM potential.

$$j^{\nu} = \frac{i}{\kappa\alpha\sqrt{-g}} \mathrm{j}^{\nu}$$

$$k = -\frac{i\alpha}{2} \rho_e$$

→ Maxwell electrodynamics

Coupling between four-velocity and nonmetricity geometrically represents electric charge density

- Imaginary nonmetricity so covariant derivative with respect to Γ coincides with covariant derivative of U(1) gauge symmetry.

- Coupling of Γ to spinors in tetrad formulation → $\alpha = \frac{1}{4e}$
- Bianchi identity → Lorentz equation of motion.

REMARKS

$$S_\nu = \frac{3}{8}\Big(-\frac{iA_\nu}{\alpha} + V^\rho{}_{\rho\nu}\Big)$$

• If U couples to torsion vector:

$$\mathcal{L}_S = \sqrt{-g}kS_\mu u^\mu \;\rightarrow\; V^\rho{}_{\mu\nu} = \frac{\kappa k}{8}(3g_{\mu\nu}u^\rho - 2u_{(\mu}\delta^\rho_{\nu)})$$

Einstein eqs. have additional term:

$$\kappa^{-1}G_{\mu\nu} = \ \dots \ + \frac{3}{64}\kappa k^2(2u_\mu u_\nu + g_{\mu\nu})$$

corresponding to fluid with $\rho_n = -\frac{\kappa n^2}{24}$ and $p = -\frac{\rho_n}{3}$

n – charge-particle concentration

$\rightarrow$ no strong energy condition at Planck scale

• EM gauge transformation related to λ-transformation of connection (that leaves Ricci tensor invariant).

• Adding projectively non-invariant constraint on torsion as Lagrange multiplier gives mass $\sqrt{\frac{3}{2\kappa\alpha}}$ to nonmetricity-trace degree of freedom (Maxwell $\rightarrow$ Proca)

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CONCLUDING REMARKS

- Mathematical exercise in classical theory, rewriting Maxwell electrodynamics in terms of geometrical quantities.

- Though QED is the correct theory of EM, a possible geometrical theory of classical EM and gravity can be regarded as the classical limit of the unified QED + QG.

- Out of 5 possible modifications of 2^{nd} pair of Maxwell eqs. for general connection, 4 contain full connection and 1 contains Christoffel symbols. Last possibility (metrically modified Maxwell eqs.) favored by experimentally confirmed conservation laws for electric charge and magnetic flux (Hehl). Geometric formulation of EM automatically leads to metrically modified Maxwell eqs.

- Quantization of EM potential may be related to quantization of affine connection (connection is fundamental as in LQG).

- New physics?

Planck scale: Dirac eq. $\rightarrow$ Heisenberg–Ivanenko eq.