Geometrization of electromagnetism in purely affine and metric-affine gravity

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2008 American Physical Society April Meeting
April 13, 2008, St. Louis, MO

3 FORMULATIONS OF GR

PURELY AFFINE (Einstein–Eddington)

- Lagrangian density L depends on torsionless affine connection Γ and symmetric part of the Ricci tensor of the connection R(Γ)
- The metric tensor g is the derivative of L with respect to R
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PURELY METRIC (Einstein–Hilbert)
- g is the dynamical variable
- Γ is the Levi-Civita connection of g
- L is linear in R

All 3 formulations of gravitation are dynamically equivalent

M. Ferraris and J. Kijowski, Gen. Relativ. Gravit. **14**, 165 (1982)

GEOMETRIZATION OF ELECTROMAGNETISM

CONFORMAL GEOMETRY (Weyl)

- Covariant derivative of metric = metric × Weyl vector
- Connection invariant under conformal transformation of metric of Weyl vector undergoes gauge transformation
- Associates EM potential with Weyl vector – unphysical

FIVE-DIMENSIONAL THEORY (Kaluza)

- Riemannian spacetime in 5D, L is linear in 5D Ricci scalar
- EM potential is part of 5D metric
- Field equations from variation of 5D metric → Einstein-Maxwell eqs.
- 5D geodesics → Lorentz force

NONSYMMETRIC METRIC & CONNECTION

- L is linear in R
- g & Γ are variables (Schrödinger) or just g (Einstein-Straus)
- Associates EM field tensor with dual of contravariant metric tensor

EM FIELD IN AFFINE CONNECTION

• Enough degrees of freedom in connection to describe gravitational and electromagnetic fields (classically)

• Purely affine formulation allows for elegant unification of both in the absence of sources

M. Ferraris and J. Kijowski, Gen. Relativ. Gravit. **14**, 37 (1982)

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Gravitational field – symmetric part of the Ricci tensor of the connection $P_{\mu\nu} = R_{(\mu\nu)}$

EM field – the tensor of **homothetic curvature** (which is a curl)

$$Q_{\mu\nu} \doteq R^{\rho}{}_{\rho\mu\nu} = \Gamma^{\ \rho}_{\rho\,\nu,\mu} - \Gamma^{\ \rho}_{\rho\,\mu,\nu}$$

V. N. Ponomarev and Y. N. Obukhov, Gen. Relativ. Gravit. **14**, 309 (1982)

We consider purely affine Lagrangians that depend on connection, and curvature via contracted curvature tensors

FIELD EQUATIONS

Metric density of purely affine gravity $\mathrm{g}^{\mu\nu} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial P_{\mu\nu}}$ (Eddington)

Metric tensor $g^{\mu\nu} \equiv \frac{\mathrm{g}^{\mu\nu}}{\sqrt{-\det \mathrm{g}^{\rho\sigma}}}$ $\rightarrow$ algebraic relation R(g)

Assume $\mathfrak{L}$ is independent of $R_{[\mu\nu]}$

Define:

$$\mathrm{h}^{\mu\nu} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial Q_{\mu\nu}} \qquad \Pi^{\mu\ \ \nu}_{\ \ \rho} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial \Gamma^{\ \rho}_{\mu\ \nu}}$$ hypermomentum (Hehl)

$${}^{*}\Gamma^{\ \rho}_{\mu\ \nu} = \Gamma^{\ \rho}_{\mu\ \nu} + \frac{2}{3}\delta^{\rho}_{\mu} S_{\nu}$$ projectively invariant

Field equations from **purely affine** variational principle:
– variation of action with respect to $\Gamma^{\ \rho}_{\mu\ \nu}$

FIELD EQUATIONS

$$
\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^{*}\Gamma^{\ \mu}_{\sigma\ \rho}\mathrm{g}^{\sigma\nu} + {}^{*}\Gamma^{\ \nu}_{\rho\ \sigma}\mathrm{g}^{\mu\sigma} - {}^{*}\Gamma^{\ \sigma}_{\sigma\ \rho}\mathrm{g}^{\mu\nu}
$$
$$
= \Pi^{\mu}{}_{\rho}{}^{\nu} - \frac{1}{3}\Pi^{\mu}{}_{\sigma}{}^{\sigma}\delta^{\nu}_{\rho} + 2\mathrm{h}^{\nu\sigma}{}_{,\sigma}\delta^{\mu}_{\rho} - \frac{2}{3}\mathrm{h}^{\mu\sigma}{}_{,\sigma}\delta^{\nu}_{\rho}
$$

gives

$$
\boxed{\mathrm{h}^{\nu\sigma}{}_{,\sigma} = \mathrm{j}^{\nu} \equiv \frac{1}{8}\Pi^{\sigma}{}_{\sigma}{}^{\nu}}
$$

← looks like 2nd pair of Maxwell eqs.

- The current density is conserved even if $\mathfrak{L}$ depends on $R_{[\mu\nu]}$
- If $\mathfrak{L}$ is independent of Q then field equations constrain how $\mathfrak{L}$ depends on Γ – unphysical – EM represented by Q **replaces** this constraint by Maxwell equations

FIELD EQUATIONS

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^*\Gamma^{\ \mu}_{\sigma\ \rho}\mathrm{g}^{\sigma\nu} + {}^*\Gamma^{\ \nu}_{\rho\ \sigma}\mathrm{g}^{\mu\sigma} - {}^*\Gamma^{\ \sigma}_{\sigma\ \rho}\mathrm{g}^{\mu\nu}$$
$$= \Pi^{\mu}{}_{\rho}{}^{\nu} - \frac{1}{3}\Pi^{\mu}{}_{\sigma}{}^{\sigma}\delta^{\nu}_{\rho} + 2\mathrm{h}^{\nu\sigma}{}_{,\sigma}\delta^{\mu}_{\rho} - \frac{2}{3}\mathrm{h}^{\mu\sigma}{}_{,\sigma}\delta^{\nu}_{\rho}$$

Einstein eqs.

↑ + algebraic R(g)

gives

Γ(g,Π) → differential R(g,Π)

$$\mathrm{h}^{\nu\sigma}{}_{,\sigma} = \mathrm{j}^{\nu} \equiv \frac{1}{8}\Pi^{\sigma}{}_{\sigma}{}^{\nu}$$

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- The current density is conserved even if $\mathfrak{L}$ depends on $R_{[\mu\nu]}$
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ANALOGY WITH CLASSICAL MECHANICS

Generalized coordinate	q^i	$\Gamma_{\mu\ \nu}^{\ \rho}$
Generalized velocity	$\dot{q}^i$	$P_{\mu\nu}$
Lagrangian	$\mathfrak{L}(\Gamma, R, Q)$	$L(q^i, \dot{q}^i)$
Canonical momentum	$p^i = \frac{\partial L}{\partial \dot{q}^i}$	$\mathfrak{g}^{\mu\nu} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial P_{\mu\nu}}$
Generalized force	$f^i = \frac{\partial L}{\partial q^i}$	$\Pi^{\mu\ \nu}_{\ \rho} \equiv -2\kappa \frac{\partial \mathfrak{L}}{\partial \Gamma_{\mu\ \nu}^{\ \rho}}$

J. Kijowski, Gen. Relativ. Gravit. **9**, 857 (1978)

EQUIVALENCE OF AFFINE AND METRIC GRAVITY

Affine Hamiltonian via **Legendre transformation** (Ferraris & Kijowski)

$$\mathfrak{H}(\Gamma, \mathrm{g}, Q) = \mathfrak{L} - \frac{\partial \mathfrak{L}}{\partial P_{\mu\nu}} P_{\mu\nu} = \mathfrak{L} + \frac{1}{2\kappa} \mathrm{g}^{\mu\nu} P_{\mu\nu} \qquad H = p_i \dot{q}_i - L$$

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1st Hamilton equation (δg)
→ Einstein equations

$$P_{\mu\nu} = 2\kappa \frac{\partial \mathfrak{H}}{\partial \mathrm{g}^{\mu\nu}} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$P_{\mu\nu} - \frac{1}{2} P g_{\mu\nu} = \kappa \Theta_{\mu\nu} \qquad 2\kappa\delta\mathcal{L} = \Theta_{\mu\nu} \delta \mathrm{g}^{\mu\nu}$$

$\mathfrak{H}$ is metric Lagrangian for matter $\mathcal{L}$

2nd Hamilton equation (δΓ) → FE

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}$$

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Metric Lagrangian for the gravitational field is a Legendre term pv
→ ***L is automatically linear in R – no f(R) models*** (NP)

Metric–Affine → Metric: Legendre transformation with respect to Γ

ELECTROMAGNETIC FIELD

Metric Maxwell Lagrangian

$$\mathfrak{H}_{EM} = -\frac{1}{4}\sqrt{-g}F_{\alpha\beta}F_{\rho\sigma}g^{\alpha\rho}g^{\beta\sigma}$$

Affine Ferraris–Kijowski Lagrangian (dynamically equivalent to Maxwell)

$$\mathfrak{L}_{EM} = -\frac{1}{4}\sqrt{-\wp}F_{\alpha\beta}F_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}$$

$\wp = \det P_{\mu\nu}$

$P^{\mu\nu}$ inverse of $P_{\mu\nu}$

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This equivalence is valid so long as there are no other terms in L depending on $R_{\mu\nu}$. Also, limit F=0 is bad, we need background field – Λ

Unified Ferraris–Kijowski Lagrangian (associate F with eQ)

$$\mathfrak{L}_{EM} = -\frac{e^2}{4}\sqrt{-\wp}Q_{\alpha\beta}Q_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}$$

$$\mathrm{h}^{\mu\nu} = \kappa e^2\sqrt{-\wp}Q_{\alpha\beta}P^{\mu\alpha}P^{\nu\beta}$$

$$S_\nu = \frac{3}{8}\left(-\frac{A_\nu}{e} + V^\rho{}_{\rho\nu}\right)$$

EM potential corresponds to trace of nonmetricity

EM LAGRANGIAN AS LEGENDRE TERM FOR Q

New Hamiltonian via Legendre transformation with respect to **homothetic curvature** (NP)

$$\mathfrak{f} = \mathfrak{P} - \frac{\partial \mathfrak{P}}{\partial Q_{\mu\nu}} Q_{\mu\nu} = \mathfrak{P} + \frac{1}{2\kappa} \mathsf{h}^{\mu\nu} Q_{\mu\nu}$$

δg $\rightarrow$ Einstein equations

δΓ $\rightarrow$ 2nd pair of Maxwell equations

δh $\rightarrow$ $Q_{\mu\nu} = 2\kappa \frac{\partial \mathfrak{f}}{\partial \mathsf{h}^{\mu\nu}}$

$$S = \frac{1}{c} \int d^4x \left(-\frac{1}{2\kappa} P\sqrt{-g} - \frac{1}{2\kappa} \mathsf{h}^{\mu\nu} Q_{\mu\nu} + \mathfrak{f}(\Gamma_{\mu}{}^{\rho}{}_{\nu}, \mathsf{g}^{\mu\nu}, \mathsf{h}^{\mu\nu}) \right)$$

Maxwell–Palatini Lagrangian (vary A and F independently)

$$\mathsf{L}_{\mathrm{EM}} = \sqrt{-g} \left(A_{\nu,\mu} F^{\mu\nu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_{\mu} j^{\mu} \right)$$

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$\rightarrow$ Legendre term with respect to Γ

*Standard EM for **simplest** (quadratic) dependence on Q (or f)*

NP, Int. J. Mod. Phys. A **23**, 567 (2008)

CONCLUDING REMARKS

• Mathematical exercise in classical theory.
• Though QED is the correct theory of EM, a possible geometrical theory of classical EM and gravity can be regarded as the classical limit of the unified QED + QG.
• Associating EM field with homothetic curvature (i.e., EM potential with nonmetricity which is part of affine connection) seems more natural than relating EM potential to metric, because both EM potential and affine connection have the same purpose: to correct ordinary derivative into covariant derivative.
• EM gauge transformation related to λ-transformation of connection.
• Quantization of EM potential may be related to quantization of affine connection (e.g., in Ashtekar formulation).
• Purely affine formulation automatically generates metric Lagrangian for gravity that is **linear** in Ricci scalar, cosmological constant as **background** field, and Maxwell electrodynamics for **simplest** dependence on **homothetic** curvature; metric Lagrangians are just Legendre terms.
• New physics? Planck scale: Dirac eq. → Heisenberg–Ivanenko eq.